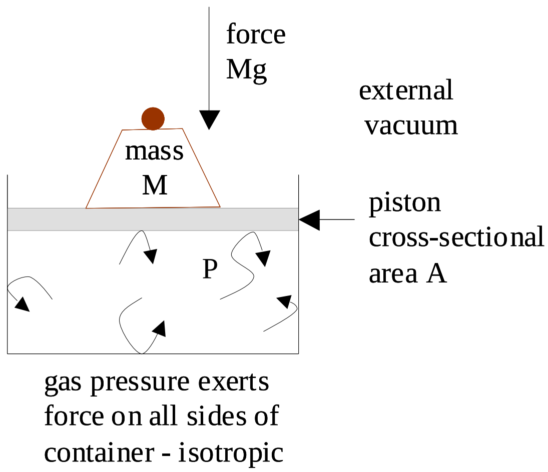
Problem

The equation of state does not itself define the reaction of the atmosphere to the variety of processes occurring in nature. For example, if we decrease the pressure of a mass of atmosphere by forcing it up a mountainside (say), we do not necessarily know how volume (or density) and temperature will change independently, we only know how their ratio changes

$$\frac{p\alpha}{T} = R$$

If P doubles, α/T decreases to half its original value, but how does T itself change? We need to know more about the process. Stay tuned!

First, we will consider how pressure decreases with height in the atmosphere. This is given by the hydrostatic equation.



Mg=pA

We can substitute the weight sitting on the piston by the mass of the entire atmosphere above the point of interest, and acted upon by gravity.

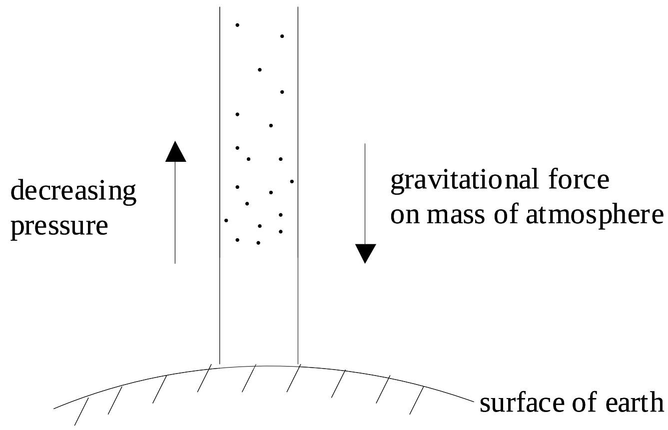
Mg=pA

P=Mg/A

Hydrostatic balance

Why clouds do not fall?

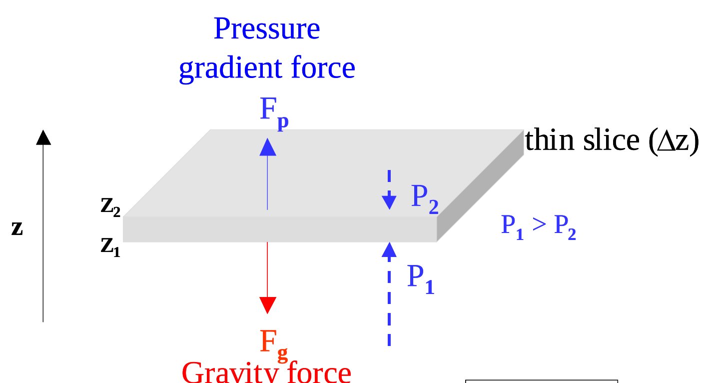
Now consider a column of atmosphere:



Decreasing pressure with height imposes an upward force on any element of fluid (the pressure gradient force), which almost exactly balances the gravitational force due to the mass of atmosphere above.

Or, the upward pressure gradient force acting on a thin slice of air (p decreases with z) is generally very closely in balance with the downward force due to gravitational force.

Note: gradient means change with distance (e.g., temperature gradient)



$$\Delta z = z_2 - z_1 > 0$$
$$\Delta p = p_2 - p_1 < 0$$

$$\boxed{F_p = F_g}$$

We can express this balance mathematically:

The net upward force (+) acting on this slice of air is:

$$F_p = -\Delta p \cdot A, \qquad \Delta p = p_2 - p_1 < 0$$

where A is the horizontal cross section area of the thin slice.

The gravitational force acting on this slice is:

$F_g = -m \cdot g$, (g = 9.81 m s^{-2})

where g is the gravitational acceleration, and m is the mass of the thin slice and it is:

$m = \rho \cdot (A \cdot \Delta z)$ (density x volume)

Pressure gradient force = gravitational force

$$F_p = -F_g$$

$$-\Delta p \cdot A = \rho \cdot A \cdot \Delta z \cdot g$$

$$\frac{1}{\rho}\frac{\Delta p}{\Delta z} = -g$$

When $\Delta z \to 0$, $\Delta p \to 0$ ($\Delta z \to 0$), the differential form is:

$$\frac{1}{\rho}\frac{dp}{dz} = -g, \text{ or}$$

$$\alpha \cdot dp = -g \cdot dz$$

where α is specific volume.

This is the hydrostatic equation and expresses the magnitude of the rate of decreases of pressure with height under the condition of hydrostatic balance. The equation is of fundamental importance in meteorology.

An alternative way of writing the hydrostatic equation is to express density in terms of temperature and pressure using the equation of state

i.e. $p = \rho RT, \quad \rho = \dfrac{p}{RT}$

So, $$\frac{dp}{dz} = -\frac{pg}{RT}$$

For small increment,

$$\frac{\Delta p}{\Delta z} \approx -\frac{pg}{RT}$$

Questions

1) Under what conditions does pressure decrease exponentially with height (as we assumed in one of our early class sessions)? Can you express the "scale height" of the atmosphere in terms of temperature?
2) How quickly does pressure decrease with height near the earth's surface?
3) Can you derive scale height using ideal gas law and hydrostatic balance equation?

Scale height and the hypsometric equation

$$\frac{dp}{dz} = -\frac{pg}{RT} = -\frac{pg}{R_d T_v}$$

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$$\int_{p_1}^{p_2} \frac{1}{p} dp = -\int_{z_1}^{z_2} \frac{g}{R_d T_v} dz$$

$$d \ln p \Big|_{p_1}^{p_2} = -\frac{g}{R_d} \int_{z_1}^{z_2} \frac{1}{T_v} dz$$

Assume T_v is a constant

$$\ln p_2 - \ln p_1 = -\frac{g}{R_d T_v}(z_2 - z_1)$$

$$\ln \frac{p_2}{p_1} = -\frac{g}{R_d T_v}(z_2 - z_1)$$

$$p_2 = p_1 \exp\left(-\frac{g}{R_d T_v}(z_2 - z_1) \right)$$

$$= p_1 \exp\left(-\frac{(z_2 - z_1)}{\frac{R_d T_v}{g}} \right)$$

$$= p_1 \exp\left(-\frac{(z_2 - z_1)}{H} \right)$$

$H = \dfrac{R_d T_v}{g}$, scale height

$z_2 - z_1 = -\dfrac{R_d T_v}{g} \ln \dfrac{p_2}{p_1} = H \ln \dfrac{p_1}{p_2}$, hypsometric equation